A SOLUTION FOR THE MOTION OF A BODY WITH A FIXED POINT

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1. If a given body is subject to the gravitational force and carries rotating masses (flywheel or a liquid circulating within multiply-connected cavities) then its motion is described by Equations [1]

$$A\frac{dp}{dt} = (B - C)\,qr + \lambda_2 r - \lambda_3 q + e_2\gamma_3 - e_3\gamma_2 \tag{1.1}$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \qquad \left(\begin{array}{c} 1\,2\,3 \\ ABC \\ pqr \end{array}\right) \tag{1.2}$$

Here e_1, e_2, e_3 is the unit vector passing from the fixed point through the center of gravity of the system; $\gamma_1, \gamma_2, \gamma_3$ is the vector in the direction of the gravitational force, the magnitude of which is equal to the product of the weight of the body and the distance connecting the center of gravity with the fixed point; $\lambda_1, \lambda_2, \lambda_3$ are constants characterizing the ciclic motions; the remaining notation is as usual [1].

The integrals of Equations (1.1) and (1.2) are

$$Ap^2 + Bq^2 + Cr^2 - 2\,(e_1\gamma_1 + e_2\gamma_2 + e_3\gamma_3) = 2E \tag{1.3}$$

$$(Ap + \lambda_1)\,\gamma_1 + (Bq + \lambda_2)\,\gamma_2 + (Cr + \lambda_3)\,\gamma_3 = k \tag{1.4}$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \Gamma^2 \tag{1.5}$$

The mass distribution within the body and internal ciclic motions are characterized by nine parameters $A, B, C, \Gamma, \lambda_1, \lambda_2, \lambda_3$ and two of the three quantities e_1, e_2, e_3, connected by the relationship

$$e_1^2 + e_2^2 + e_3^2 = 1 \tag{1.6}$$

The general solution containing 15 independent parameters (here are included the six parameters characterizing the initial conditions) is unknown, and therefore the generality of each particular solution will be determined by the number of independent parameters retained in it. Zhukovskii [1] gave a solution of Equations (1.1) by constraining the system parameters by three conditions, namely, he assumed the center of gravity coincident with the point of support. His solution, consequently, has 12 independent parameters. With the three additional conditions $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the Zhukovskii solution yields the Euler solution containing 9 parameters.

With the five conditions $B = C$, $\lambda_2 = \lambda_3 = 0$, $e_2 = e_3 = 0$ we obtain from

(1.1) and (1.2) the Lagrange solution, containing 10 parameters.

Recently Sretenskii showed two more solutions [2] and [3]. The first Sretenskii solution generalizes the Goriachev-Chaplygin solution and contains 8 parameters, while the second generalizes the Hess solution and contains 11 parameters. The present study gives a solution of the problem cosidered which contains 10 independent parameters.

2. Let the center of gravity be in one of the princilal planes of the changed momental ellipsoid $e_2 = 0$, then in view of (1)

$$e_1 = \cos\alpha, \qquad e_3 = \sin\alpha \tag{2.1}$$

Leaving the parameter λ_2 arbitrary (subscript 2 is omitted in the following), we subject λ_1 and λ_3 to condition

$$(2B - C)\lambda_1 \sin\alpha = (2B - A)\lambda_3 \cos\alpha \tag{2.2}$$

By means of a new parameter ν this condition is represented in the form

$$\lambda_1 = (2B - A)\nu\cos\alpha, \qquad \lambda_3 = (2B - C)\nu\sin\alpha \tag{2.3}$$

We will seek a solution for which the two components of angular velocity are constant $p = p_0$ and $r = r_0$. At the same time the first and third equation of (1.1) establish the dependence of γ_2 on q if

$$p_0 = \nu\cos\alpha, \qquad r_0 = \nu\sin\alpha \tag{2.4}$$

and this dependence is

$$\gamma_2 = \nu(\lambda - Bq) \tag{2.5}$$

As a consequence of (2.1) to (2.5) the integrals (1.3) and (1.5) yield

$$\gamma_1 = (H + {}^1/_2 Bq^2)\cos\alpha - \sqrt{\Gamma^2 - (H + {}^1/_2 Bq^2)^2 - \nu^2(\lambda - Bq)^2}\,\sin\alpha \tag{2.6}$$
$$\gamma_3 = (H + {}^1/_2 Bq^2)\sin\alpha + \sqrt{\Gamma^2 - (H + {}^1/_2 Bq^2)^2 - \nu^2(\lambda - Bq)^2}\,\cos\alpha$$
$$H = {}^1/_2\nu^2(A\cos^2\alpha + C\sin^2\alpha) - E$$

By substituting Expressions (2.1), (2.3), (2.4) and (2.6) into the second equation of (1.1), q is definded as an elliptic function of time

$$t = -B\int_{q_0}^{q} \frac{dq}{\sqrt{\Gamma^2 - (H + {}^1/_2 Bq^2)^2 - \nu^2(\lambda - Bq)^2}} \tag{2.7}$$

The derived quantities satisfy Equations (1.2) and by substituting them into (1.4) they determine the constant $k = \nu(BH + \lambda^2)$.

The nutation and rotation angles are found from Formulas

$$\gamma_1 = \Gamma\sin\theta\cos\varphi, \qquad \gamma_2 = \Gamma\cos\theta, \qquad \gamma_3 = \Gamma\sin\theta\sin\varphi$$

For the angle of precession we have

$$\frac{d\psi}{dt} = \Gamma\frac{p\gamma_1 + r\gamma_3}{\gamma_1^2 + \gamma_3^2}, \qquad \psi = \psi_0 + \Gamma\int_0^t \frac{H + {}^1/_2 Bq^2(\tau)}{\Gamma^2 - \nu^2[\lambda - Bq(\tau)]^2}\,d\tau$$

Thus, in the derived solution the independent parameters $A, B, C, \Gamma, \alpha, \lambda, \nu, H, q_0$ and ψ_0 are retained.

The particular case of the derived solution for the condition $\alpha = 0$ is given in [4].

Two known solutions for the motion of a rigid body with a fixed point follow from the found solution in cases of $\lambda_i = 0$ $(i = 1,2,3)$.

1. Let

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \text{при} \ \nu \neq 0 \tag{2.8}$$

We conclude from (2.3) that these conditions are fulfilled if

$$\alpha = 0, \qquad A = 2B \tag{2.9}$$

Under conditions (2.8) and (2.9), we find from (2.4) to (2.7) that

$$p = \nu, \qquad r = 0, \qquad \gamma_1 = H + \frac{B}{2} q^2, \qquad \gamma_2 = -B\nu q$$

$$\gamma_3 = \sqrt{\Gamma^2 - \nu^2 B^2 q^2 - (H + {}^1/_2 Bq^2)^2}, \qquad t = -B \int_{q_0}^{q} \frac{dq}{\sqrt{\Gamma^2 - \nu^2 B^2 q^2 - (H + {}^1/_2 Bq^2)^2}}$$

which coincides with the Bobylev [5] – Steklov [6] solution. The latter one contains seven independent parameters.

2. Let $\nu = 0$, then from (2.3), (2.4) and (2.5)

$$p = r = 0, \qquad \gamma_2 = 0, \qquad \lambda_1 = \lambda_3 \Rightarrow 0 \tag{2.10}$$

i.e. the axis of rotation is fixed and horizontal. From (2.8), (2.9) and (2.10) we have

$$\theta = {}^1/_2\pi, \qquad \psi = \psi_0, \qquad \gamma_1 = \Gamma\cos\varphi, \qquad \gamma_3 = \Gamma\sin\varphi$$

One of the Euler kinematic equations yields $q = d\varphi/dt$. Substituting these values into the second equation in (1.1), we obtain the case of the physical pendulum.

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